

# Asymptotic velocity of a position-dependent quantum walk

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## Abstract

We consider a position-dependent coined quantum walk on  $\mathbb{Z}$  and assume that the coin operator  $C(x)$  satisfies

$$\|C(x) - C_0\| \leq c_1|x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}$$

with positive  $c_1$  and  $\epsilon$  and  $C_0 \in U(2)$ . We show that the Heisenberg operator  $\hat{x}(t)$  of the position operator converges to the asymptotic velocity operator  $\hat{v}_+$  so that

$$\text{s-}\lim_{t \rightarrow \infty} \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) = \Pi_p(U) + \exp(i\xi \hat{v}_+) \Pi_{ac}(U)$$

provided that  $U$  has no singular continuous spectrum. Here  $\Pi_p(U)$  (resp.  $\Pi_{ac}(U)$ ) is the orthogonal projection onto the direct sum of all eigenspaces (resp. the subspace of absolute continuity) of  $U$ . We also prove that for the random variable  $X_t$  denoting the position of a quantum walker at time  $t \in \mathbb{N}$ ,  $X_t/t$  converges in law to a random variable  $V$  with the probability distribution

$$\mu_V = \|\Pi_p(U)\Psi_0\|^2 \delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{ac}(U)\Psi_0\|^2,$$

where  $\Psi_0$  is the initial state,  $\delta_0$  the Dirac measure at zero, and  $E_{\hat{v}_+}$  the spectral measure of  $\hat{v}_+$ .

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# 1 Introduction

The weak limit theorems for discrete time quantum walks have been studied in various models (for reviews, see [7, 12]). In his papers [5, 6], Konno first proved the weak limit theorem for a position-independent quantum walk on  $\mathbb{Z}$ . Grimmett et al [4] simplified the proof and extended the result to higher dimensions. For position-dependent quantum walks on  $\mathbb{Z}$ , the weak limit theorems were obtained by Konno et al [9], Endo and Konno [2], and Endo et al [3].

We consider a position-dependent quantum walk on  $\mathbb{Z}$  given by a unitary evolution operator  $U$ :

$$(U\Psi)(x) = P(x+1)\Psi(x+1) + Q(x-1)\Psi(x-1), \quad x \in \mathbb{Z},$$

where  $\Psi$  is a state vector in the Hilbert space  $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2)$  of states and

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.$$

Let  $C(x) = P(x) + Q(x) \in U(2)$  and  $S$  be a shift operator such that  $U = SC$ . Suppose that there exists a unitary matrix  $C_0 = P_0 + Q_0 \in U(2)$  such that

$$\|C(x) - C_0\| \leq c_1|x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\} \quad (1.1)$$

with positive  $c_1$  and  $\epsilon$  independent of  $x$ . Here  $\|M\|$  stands for the operator norm of a matrix  $M \in M_2(\mathbb{C})$ . A typical example is the quantum walks with one defect [1, 8, 9, 13], which clearly satisfies (1.1). We note that the condition (1.1) allows not only finite but also infinite defects, whereas the models introduced in [2, 3] do not satisfy (1.1). The unitary operator  $U_0 = SC_0$  also defines an evolution of a position-independent quantum walk on  $\mathbb{Z}$  and satisfies

$$(U_0\Psi)(x) = P_0\Psi(x+1) + Q_0\Psi(x-1), \quad x \in \mathbb{Z}$$

with  $C_0 = P_0 + Q_0$ . Let  $\hat{x}$  be the position operator defined by  $(\hat{x}\Psi)(x) = x\Psi(x)$ ,  $x \in \mathbb{Z}$ . and  $\hat{x}_0(t) = U_0^{-t}\hat{x}U_0^t$  the Heisenberg operator of  $\hat{x}$  at time  $t \in \mathbb{N}$  with the evolution  $U_0$ . In [4], Grimmett et al essentially proved that the operator  $\hat{x}_0(t)/t$  weakly converges to the asymptotic velocity operator  $\hat{v}_0$  so that

$$\text{w-}\lim_{t \rightarrow \infty} \exp\left(i\xi \frac{\hat{x}_0(t)}{t}\right) = \exp(i\xi \hat{v}_0), \quad \xi \in \mathbb{R}. \quad (1.2)$$

Let  $X_t^{(0)}$  be the random variable denoting the position of a quantum walker at time  $t \in \mathbb{N}$  with the evolution operator  $U_0$ . Then, the characteristic function

of  $X_t^{(0)}/t$  is given by

$$\mathbb{E}(e^{i\xi X_t^{(0)}/t}) = \langle \Psi_0, e^{i\xi \hat{x}_0(t)/t} \Psi_0 \rangle, \quad \xi \in \mathbb{R},$$

where  $\Psi_0$  is the initial state of the quantum walker. Hence, (1.2) means that the random variable  $X_t^{(0)}/t$  converges in law to a random variable  $V_0$ , which represents the linear spreading of the quantum walk:  $X_t^{(0)} \sim tV_0$ .

In this paper, we derive the asymptotic velocity  $\hat{v}_+$  for the Heisenberg operator  $\hat{x}(t) = U^{-t}\hat{x}U^t$  with the evolution  $U$  of the position-dependent quantum walk. The decaying condition (1.1) implies that  $U - U_0$  is a trace class operator and allows us to prove the existence and completeness of the wave operator

$$W_+ = \text{s-}\lim_{t \rightarrow \infty} U^{-t}U_0^t\Pi_{\text{ac}}(U_0)$$

using a discrete analogue of the Kato–Rosenblum Theorem (See [11] for details), where  $\Pi_{\text{ac}}(U_0)$  is the orthogonal projection onto the subspace of absolute continuity of  $U_0$ . We also prove that

$$\text{s-}\lim_{t \rightarrow \infty} \exp\left(i\xi \frac{\hat{x}_0(t)}{t}\right) = \exp(i\xi \hat{v}_0), \quad \xi \in \mathbb{R}$$

under a reasonable condition, which is essentially the same as that of [4]. Furthermore, we assume that  $U$  has no singular continuous spectrum. Then, we prove that

$$\text{s-}\lim_{t \rightarrow \infty} \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) = \Pi_{\text{p}}(U) + \exp(i\xi \hat{v}_+)\Pi_{\text{ac}}(U), \quad (1.3)$$

where  $\Pi_{\text{p}}(U)$  is the orthogonal projection onto the direct sum of all eigenspaces of  $U$  and  $\hat{v}_+ = W_+\hat{v}_0W_+^*$ . We believe that the absence of a singular continuous spectrum can be checked with a concrete example such as the one-defect model. As a consequence of (1.3), we have the following weak limit theorem. Let  $X_t$  be the random variable denoting the position of a quantum walker at time  $t \in \mathbb{N}$  with the evolution operator  $U$  and the initial state  $\Psi_0$ . We prove that  $X_t/t$  converges in law to a random variable  $V$  with a probability distribution

$$\mu_V = \|\Pi_{\text{p}}(U)\Psi_0\|^2\delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{\text{ac}}(U)\Psi_0\|^2,$$

where  $\delta_0$  is the Dirac measure at zero and  $E_{\hat{v}_+}$  the spectral measure of  $\hat{v}_+$ .

The remainder of this paper is organized as follows. In Section 2, we present the precise definition of the model and our results. Section 3 is devoted to the proof of the existence and completeness of the wave operator. In Section 4, we construct the asymptotic velocity.

## 2 Definition of the model

Let  $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2)$  be the Hilbert space of the square-summable functions  $\Psi : \mathbb{Z} \rightarrow \mathbb{C}^2$ . We define a shift operator  $S$  and a coin operator  $C$  on  $\mathcal{H}$  as follows. For a vector  $\Psi = \begin{pmatrix} \Psi^{(0)} \\ \Psi^{(1)} \end{pmatrix} \in \mathcal{H}$ ,  $S\Psi$  is given by

$$(S\Psi)(x) = \begin{pmatrix} \Psi^{(0)}(x+1) \\ \Psi^{(1)}(x-1) \end{pmatrix}, \quad x \in \mathbb{Z}.$$

Let  $\{C(x)\}_{x \in \mathbb{Z}} \subset U(2)$  be a family of unitary matrices with

$$C(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

$C\Psi$  is given by

$$(C\Psi)(x) = C(x)\Psi(x), \quad x \in \mathbb{Z}.$$

We define an evolution operator as  $U = SC$ .  $U$  satisfies

$$(U\Psi)(x) = P(x+1)\Psi(x+1) + Q(x-1)\Psi(x-1), \quad x \in \mathbb{Z}$$

with

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.$$

For a matrix  $M \in M(2, \mathbb{C})$ , we use  $\|M\|$  to denote the operator norm in  $\mathbb{C}^2$ :  $\|M\| = \sup_{\|\mathbf{x}\|_{\mathbb{C}^2}=1} \|M\mathbf{x}\|_{\mathbb{C}^2}$ . We suppose that:

(A.1) There exists a unitary matrix  $C_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in U(2)$  such that

$$\|C(x) - C_0\| \leq c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}$$

with some positive  $c_1$  and  $\epsilon$  independent of  $x$ .

We denote by  $\mathcal{T}_1$  the set of trace class operators.

**Lemma 2.1.** Let  $U$  satisfy (A.1) and set  $U_0 = SC_0$ . Then,  $U - U_0 \in \mathcal{T}_1$ .

*Proof.* Let  $T = U - U_0$  and  $T(x) = C(x) - C_0$ . Then

$$T^*T = (C - C_0)^*(C - C_0) \tag{2.1}$$

is the multiplication operator by the matrix-valued function  $T(x)^*T(x)$ . Let  $t_i(x)$  ( $i = 1, 2$ ) be the eigenvalues of the Hermitian matrix  $T(x)^*T(x) \in$

$M(2, \mathbb{C})$  and take an orthonormal basis (ONB)  $\{\tau_i(x)\}_{i=1,2}$  of corresponding eigenvectors for all  $x \in \mathbb{Z}$ . We use  $|\xi\rangle\langle\eta|$  to denote the operator on  $\mathcal{H}$  defined by  $|\xi\rangle\langle\eta|\Psi = \langle\eta, \Psi\rangle\xi$ . Then, we have

$$T^*T = \sum_{i=1,2} \sum_{x \in \mathbb{Z}} t_i(x) |\tau_{i,x}\rangle\langle\tau_{i,x}|, \quad (2.2)$$

where  $\{\tau_{i,x}\}$  is the ONB given by

$$\tau_{i,x}(y) = \delta_{xy} \tau_i(x), \quad y \in \mathbb{Z}.$$

Since  $T^*(x)T(x) \geq 0$ , we have  $t_i(x) \geq 0$ . By (A.1), we know that

$$\max_{i=1,2} t_i(x) \leq c_1^2 |x|^{-2-2\epsilon}.$$

Hence, we have

$$\mathrm{Tr}|T| = \sum_{x \in \mathbb{Z}} \sum_{i=1,2} t_i(x)^{1/2} \leq 2c_1 \sum_{x \in \mathbb{Z}} |x|^{-1-\epsilon} < \infty,$$

which means that  $T \in \mathcal{T}_1$ . Since  $\mathcal{T}_1$  is an ideal,  $U - U_0 = ST \in \mathcal{T}_1$ .  $\square$

**Example 2.1** (one-defect model). Let  $C_0, C'_0 \in U(2)$  be unitary matrices with  $C_0 \neq C'_0$  and set

$$C(x) = \begin{cases} C'_0, & x = 0 \\ C_0, & x \neq 0. \end{cases}$$

$U = SC$  satisfies (A.1), because  $C(x) - C_0 = 0$  if  $x \neq 0$ .

**Example 2.2.** Let  $C_0 \in U(2)$  be a unitary matrix and  $\{C(x)\} \subset U(2)$  a family of unitary matrices. Assume that

$$\max_{i,j} |(C(x) - C_0)_{ij}| \leq c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\},$$

where  $M_{ij}$  denotes the  $ij$ -component of a matrix  $M$ . Then,  $U = SC$  satisfies (A.1), because all norm on a finite dimensional vector space are equivalent.

We prove the following theorem in Section 3 using a discrete analogue of the Kato–Rosenblum theorem.

**Theorem 2.1.** Let  $U$  and  $U_0$  be as above and assume that (A.1) holds. Then

$$W_+ = \mathrm{s-}\lim_{t \rightarrow \infty} U^{-t} U_0^t \Pi_{\mathrm{ac}}(U_0)$$

exists and is complete.

In what follows, we introduce the asymptotic velocity  $\hat{v}_0$ , obtained first in [4], of the quantum walk with the evolution  $U_0$  as follows. Let

$$\hat{U}_0(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} C_0, \quad k \in [0, 2\pi).$$

Since  $\hat{U}_0(k) \in U(2)$ ,  $\hat{U}_0(k)$  is represented as

$$\hat{U}_0(k) = \sum_{i=1,2} \lambda_i(k) |u_j(k)\rangle \langle u_j(k)|,$$

where  $\lambda_j(k)$  is an eigenvalue of  $\hat{U}_0(k)$  and  $u_j(k)$  is the corresponding eigenvector with  $\|u_j(k)\| = 1$ . The function  $k \mapsto e^{ik}$  is analytic, and so is  $\lambda_j(k)$ . We need the following assumption on  $u_j(k)$ :

(A.2) The functions  $k \mapsto u_j(k)$  are continuously differentiable in  $k$  with

$$\sup_{k \in [0, 2\pi)} \left\| \frac{d}{dk} u_j(k) \right\|_{\mathbb{C}^2} < \infty.$$

Let  $\mathcal{K}$  be the Hilbert space of square integrable functions  $f : [0, 2\pi) \rightarrow \mathbb{C}^2$  with norm

$$\|f\|_{\mathcal{K}} = \left( \int_0^{2\pi} \frac{dk}{2\pi} \|f(k)\|_{\mathbb{C}^2}^2 \right)^{1/2}.$$

Let  $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{K}$  be the discrete Fourier transform given by

$$(\mathcal{F}\Psi)(k) = \sum_{x \in \mathbb{Z}} e^{-ik \cdot x} \Psi(x), \quad \Psi \in \mathcal{H}.$$

We also use  $\hat{\Psi}(k) = \begin{pmatrix} \hat{\Psi}^{(0)}(k) \\ \hat{\Psi}^{(1)}(k) \end{pmatrix}$  to denote the Fourier transform of  $\Psi$ . The asymptotic velocity  $\hat{v}_0$  is the self-adjoint operator defined by

$$\hat{v}_0 = \mathcal{F}^{-1} \left( \int_{[0, 2\pi)}^{\oplus} \frac{dk}{2\pi} \sum_{j=1,2} \left( \frac{i\lambda'_j(k)}{\lambda_j(k)} \right) |u_j(k)\rangle \langle u_j(k)| \right) \mathcal{F}$$

The position operator  $\hat{x}$  is a self-adjoint operator defined by

$$(\hat{x}\Psi)(x) = x\Psi(x), \quad x \in \mathbb{Z}$$

with domain

$$D(\hat{x}) = \left\{ \Psi \in \mathcal{H} \mid \sum_{x \in \mathbb{Z}} |x|^2 \|\Psi(x)\|_{\mathbb{C}^2}^2 < \infty \right\}.$$

Let  $\hat{x}_0(t) = U_0^{-t} \hat{x} U_0^t$  be the Heisenberg operator of  $\hat{x}$  for the evolution  $U_0$ .

**Theorem 2.2.** Let  $\hat{v}_0$  and  $\hat{x}_0$  be as above. Suppose that (A.2) holds. Then,

$$\text{s-}\lim_{t \rightarrow \infty} \exp \left( i\xi \frac{\hat{x}_0(t)}{t} \right) = \exp(i\xi \hat{v}_0), \quad \xi \in \mathbb{R}. \quad (2.3)$$

*Proof.* By [10, Theorem VIII.21], (2.3) holds if and only if

$$\text{s-}\lim_{t \rightarrow \infty} \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} = (\hat{v}_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

which is proved in Subsection 4.1.  $\square$

**Example 2.3.** (i) Let  $C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then,  $\hat{U}_0(k)$  has eigenvalues 1 and  $-1$ , which are independent of  $k$ . By definition,  $\hat{v}_0 = 0$ . Hence, the random variable  $X_t^{(0)}/t$  converges in law to a random variable  $V_0$  with a probability distribution  $\delta_0$ .

(ii) Let  $C_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .  $\hat{U}_0(k)$  has eigenvalues  $e^{ik}$  and  $-e^{-ik}$ . Hence,  $\hat{v}_0$  has eigenvalues  $-1$  and  $1$ . The random variable  $X_t^{(0)}/t$  converges in law to a random variable  $V_0$  with a probability distribution  $\|\Psi^{(0)}\|^2 \delta_{-1} + \|\Psi^{(1)}\|^2 \delta_1$ .

(iii) Let  $C_0$  be the Hadamard matrix. The eigenvalues of  $\hat{U}_0(k)$  are given by  $\lambda_j(k) = ((-1)^j w(k) + i \sin k) / \sqrt{2}$  ( $j = 1, 2$ ), where  $w(k) = \sqrt{1 + \cos^2 k}$ . Hence,  $\hat{v}_0$  has no eigenvalue. The corresponding eigenvectors

$$u_j(k) = \sqrt{\frac{w(k) + (-1)^j \cos k}{2w(k)}} \begin{pmatrix} e^{ik} \\ (-1)^j w(k) - \cos k \end{pmatrix}$$

form an ONB of  $\mathbb{C}^2$  and satisfy (A.2). The random variable  $X_t^{(0)}/t$  converges in law to a random variable  $V_0$  with a probability distribution  $\|E_{\hat{v}_0}(\cdot)\Psi_0\|^2$ , where  $E_{\hat{v}_0}$  is the spectral measure of  $\hat{v}_0$ . Let us consider the Hadmard walk starting from the origin. Let the initial state  $\Psi_0$  satisfy  $\Psi_0(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  ( $|\alpha|^2 + |\beta|^2 = 1$ ) and  $\Psi(x) = 0$  if  $x \neq 0$ . Then,

$$d\|E_{\hat{v}_0}(v)\Psi_0\|^2 = (1 - c_{\alpha,\beta}v) f_K \left( v; \frac{1}{\sqrt{2}} \right) dv,$$

where  $c_{\alpha,\beta} = |\alpha|^2 - |\beta|^2 + \alpha\bar{\beta} + \bar{\alpha}\beta$ ,

$$f_K(v; r) = \frac{\sqrt{1-r^2}}{\pi(1-v^2)\sqrt{r^2-v^2}} I_{(-r,r)}(v)$$

is the Konno function, and  $I_A$  is the indicator function of a set  $A$ . For more details, the reader can consult [4, 7].

Let  $\hat{x}(t) = U^{-t}\hat{x}U$  be the Heisenberg operator of  $\hat{x}$  and define the asymptotic velocity  $\hat{v}_+$  for the evolution  $U$  by

$$\hat{v}_+ = W_+\hat{v}_0W_+^*.$$

We need the following assumption:

(A.3) The singular continuous spectrum of  $U$  is empty.

We are now in a position to state our main result, which is proved in Subsection 4.2.

**Theorem 2.3.** Let  $\hat{x}(t)$  and  $\hat{v}_+$  be as above. Suppose that (A.1) - (A.3) hold. Then,

$$\text{s-}\lim_{t \rightarrow \infty} \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) = \Pi_p(U) + \exp(i\xi \hat{v}_+) \Pi_{ac}(U), \quad \xi \in \mathbb{R}.$$

Let  $X_t$  be the random variable denoting the position of the walker at time  $t \in \mathbb{N}$  with the initial state  $\Psi_0$ . We use  $\Pi_p(U)$  to denote the orthogonal projection onto the direct sum of all eigenspaces of  $U$  and  $E_A$  to denote the spectral projection of a self-adjoint operator  $A$ .

**Corollary 2.4.** Let  $X_t$  be as above. Suppose that (A.1) - (A.3) hold. Then,  $X_t/t$  converges in law to a random variable  $V$  with a probability distribution

$$\mu_V = \|\Pi_p(U)\Psi_0\|^2 \delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{ac}(U)\Psi_0\|^2,$$

where  $\delta_0$  is the Dirac measure at zero.

*Proof.* From Theorem 2.1,  $\text{s-}\lim_{t \rightarrow \infty} U_0^{-t}U^t \Pi_{ac}(U)$  exists and is equal to  $W_+^*$ . Then,  $W_+$  is unitary from  $\text{Ran}W_+^* = \text{Ran}\Pi_{ac}(U_0)$  to  $\text{Ran}W_+ = \text{Ran}\Pi_{ac}(U)$ . Since, by Lemma 4.1,  $U_0$  is strongly commuting with  $\hat{v}_0$ , we know, from the intertwining property  $UW_+ = W_+U_0$ , that  $U$  is also strongly commuting with  $\hat{v}_+$ . Hence,  $\hat{v}_+$  is strongly commuting with  $\Pi_{ac}(U)$  and  $e^{i\xi \hat{v}_+} \Pi_{ac}(U) = \Pi_{ac}(U) e^{i\xi \hat{v}_+}$ . Hence, by Theorem 2.3,  $\exp(i\xi \hat{x}(t)/t)\Psi_0$  converges strongly to  $\Pi_p(U)\Psi_0 + e^{i\xi \hat{v}_+} \Pi_{ac}(U)\Psi_0$  and

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}(e^{i\xi X_t/t}) &= \langle \Psi_0, \Pi_p(U)\Psi_0 + e^{i\xi \hat{v}_+} \Pi_{ac}(U)\Psi_0 \rangle \\ &= \|\Pi_p(U)\Psi_0\|^2 + \int_{-\infty}^{\infty} e^{i\xi v} d\|E_{\hat{v}_+}(v)\Pi_{ac}(U)\Psi_0\|^2 \\ &= \int_{-\infty}^{\infty} e^{i\xi v} d\mu_V(v), \end{aligned}$$

which proves the corollary.  $\square$



**Example 2.4.** Let  $C_0$  be the Hadamard matrix and  $C(x)$  satisfy (A.1). As seen in Example 2.3 (iii), (A.2) is satisfied and the spectrum of  $U_0$  is purely absolutely continuous. Let  $\Psi_+ \in \mathcal{H}$  satisfy  $\Psi_+(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  ( $|\alpha|^2 + |\beta|^2 = 1$ ) and  $\Psi_+(x) = 0$  if  $x \neq 0$ . By Example 2.3,

$$d\|E_{\hat{v}_+}(v)\Pi_{\text{ac}}(U)W_+\Psi_+\|^2 = d\|E_{\hat{v}_0}(v)\Psi_+\|^2 = (1 - c_{\alpha,\beta}v)f_K\left(v; \frac{1}{\sqrt{2}}\right)dv.$$

Let  $\Psi_p \in \text{Ran}\Pi_p(U_0)$  be a unit vector and take the initial state  $\Psi_0$  as  $\Psi_0 = C_1\Psi_p + C_2W_+\Psi_+$  ( $|C_1|^2 + |C_2|^2 = 1$ ). Suppose that  $U = SC$  satisfies (A.3). By Corollary 2.4,  $X_t/t$  converges in law to  $V$  with a probability distribution  $\mu_V$  and

$$\mu_V(dv) = |C_1|^2\delta_0(dv) + |C_2|^2(1 - c_{\alpha,\beta}v)f_K\left(v; \frac{1}{\sqrt{2}}\right)dv.$$

### 3 Wave operator

To prove Theorem 2.1, we use the following general proposition:

**Proposition 3.1.** Let  $U$  and  $U_0$  be unitary operators on a Hilbert space  $\mathcal{H}$  and suppose that  $U - U_0 \in \mathcal{T}_1$ . The following limit exists:

$$W_+ = \text{s-}\lim_{t \rightarrow \infty} U^{-t}U_0^t\Pi_{\text{ac}}(U_0)$$

*Proof of Theorem 2.1.* Since, by Lemma 2.1,  $U - U_0 \in \mathcal{T}_1$ , the wave operator  $W_+$  exists. If we interchange the roles of  $U$  and  $U_0$ , then the proposition says that the limit  $\text{s-}\lim_{t \rightarrow \infty} U_0^{-t}U^t\Pi_{\text{ac}}(U)$  also exists, which implies that  $W_+$  is complete. This completes the proof.  $\square$

In the remainder of this section, we suppose that  $U - U_0 \in \mathcal{T}_1$  and prove Proposition 3.1. This is done by a discrete analogue of [11, Theorem 6.2.]. We use  $\mathcal{H}_{\text{ac}}$  and  $\mathcal{H}_p$  to denote the subspaces of absolute continuity and the direct sum of all eigenspaces of  $U_0$ . Let  $E_0$  be the spectral measure of  $U_0$  with  $E_0([0, 2\pi)) = I$ . Let

$$\mathcal{H}_{\text{ac},0} = \{\psi \in \mathcal{H}_{\text{ac}} \mid d\|E_0(\lambda)\psi\|^2 = G_\psi(\lambda)^2 d\lambda \text{ and } G_\psi \in L^2 \cap L^\infty\},$$

where  $L^2 = L^2([0, 2\pi))$  and  $L^\infty = L^\infty([0, 2\pi))$ . Although the following lemma may be well known, we give proofs for completeness.

**Lemma 3.1.**  $\mathcal{H}_{\text{ac},0}$  is dense in  $\mathcal{H}_{\text{ac}}$ .

*Proof.* For all  $\psi \in \mathcal{H}_{ac}$ , there exists a positive function  $F \in L^1$  such that  $d\|E_0(\lambda)\psi\|^2 = F(\lambda)d\lambda$ . Let  $B_n = F^{-1}([0, n])$ , and let  $\chi_{B_n}$  be the characteristic function of  $B_n$ . We set  $G_n = \sqrt{F}\chi_{B_n}$  and  $\psi_n = E_0(B_n)\psi$ . Then  $G_n \in L^2 \cap L^\infty$  and  $\|E_0(B)\psi\|^2 = \int_B G_n(\lambda)^2 d\lambda$ . Hence,  $\psi_n \in \mathcal{H}_{ac,0}$  and  $\psi = \lim_n \psi_n$ . This completes the proof.  $\square$

**Lemma 3.2.** Let  $\phi \in \mathcal{H}$  and  $\psi \in \mathcal{H}_{ac,0}$ . Then,

$$\sum_{t \in \mathbb{Z}} |\langle \phi, U_0^t \psi \rangle|^2 \leq 2\pi \|\phi\|^2 \sup_{\lambda} G_{\psi}(\lambda)^2.$$

*Proof.* Let  $\psi \in \mathcal{H}_{ac,0}$  and  $\mathcal{L} = L^2([0, 2\pi], G_{\psi}^2(\lambda)d\lambda)$ . Let  $H_0$  be the self-adjoint operator defined by  $\langle \xi, H_0 \eta \rangle = \int_0^{2\pi} \lambda d\langle \xi, E_0(\lambda) \eta \rangle$  ( $\xi, \eta \in \mathcal{H}$ ). Let  $\mathcal{U} : \mathcal{L} \rightarrow \mathcal{H}$  be an injection defined by  $\mathcal{U}f = f(H_0)\psi$  ( $f \in \mathcal{L}$ ). Then  $\mathcal{U}1 = \psi$  and  $\mathcal{U}e^{it\lambda} = U_0^t \psi$  ( $t \in \mathbb{N}$ ). We use  $\Pi$  to denote the orthogonal projection onto  $U\mathcal{L}$ . Let  $\phi \in \mathcal{H}$  and  $F = \mathcal{U}^{-1}\Pi\phi \in \mathcal{L}$ . Then we have

$$\langle \phi, U_0^t \psi \rangle = \int_0^{2\pi} e^{it\lambda} \bar{F}(\lambda) G_{\psi}(\lambda)^2 d\lambda = 2\pi \widehat{\bar{F}G_{\psi}^2}(t).$$

Hence, by Parseval's identity, we obtain

$$\begin{aligned} \sum_{t \in \mathbb{Z}} |\langle \phi, U_0^t \psi \rangle|^2 &= 2\pi \int_0^{2\pi} |\bar{F}(\lambda) G_{\psi}(\lambda)^2|^2 d\lambda \\ &\leq 2\pi \sup_{\lambda} G_{\psi}(\lambda)^2 \int_0^{2\pi} |\bar{F}(\lambda)|^2 G_{\psi}(\lambda)^2 d\lambda \leq 2\pi \sup_{\lambda} G_{\psi}(\lambda)^2 \|\Pi\phi\|^2. \end{aligned}$$

This completes the proof.  $\square$

Let  $W_t = U^{-t}U_0^t$ .

**Lemma 3.3.** Let  $t, s \in \mathbb{N}$  ( $s \neq t$ ). Then,  $s\text{-}\lim_{r \rightarrow \infty} (W_t - W_s)U_0^r \Pi_{ac}(U_0) = 0$ .

*Proof.* For  $t, s \in \mathbb{N}$  ( $t > s$ ), we have  $W_t = \sum_{k=s+1}^t (W_k - W_{k-1}) + W_s$  and  $W_k - W_{k-1} = U^{-k}(-T)U_0^{k-1}$ , where  $T = U - U_0 \in \mathcal{T}_1$ . Since  $\mathcal{T}_1$  is an ideal, we know that

$$W_t - W_s = \sum_{k=s+1}^t U^{-k}(-T)U_0^{k-1} \in \mathcal{T}_1.$$

In particular,  $W_t - W_s$  is compact. Let  $H_0$  be the self-adjoint operator defined in the proof of Lemma 3.2. Since  $w\text{-}\lim_{r \rightarrow \infty} e^{irH_0} \Pi_{ac}(H_0) = 0$ , we have

$$s\text{-}\lim_{r \rightarrow \infty} (W_t - W_s)U_0^r \Pi_{ac}(U_0) = s\text{-}\lim_{r \rightarrow \infty} (W_t - W_s)e^{irH_0} \Pi_{ac}(H_0) = 0.$$

This completes the proof.  $\square$

*Proof of Proposition 3.1.* By Lemma 3.1, it suffices to prove that, for  $\psi \in \mathcal{H}_{\text{ac},0}$ ,

$$\|(W_t - W_s)\psi\| \rightarrow 0, \quad t, s \rightarrow \infty.$$

Because

$$\|(W_t - W_s)\psi\|^2 = \langle \psi, W_t^*(W_t - W_s)\psi \rangle - \langle \psi, W_s^*(W_t - W_s)\psi \rangle,$$

we need only to prove that

$$\langle \psi, W_t^*(W_t - W_s)\psi \rangle \rightarrow 0, \quad t, s \rightarrow \infty.$$

By direct calculation, we have, for  $r > 1$ ,

$$\begin{aligned} & W_t^*(W_t - W_s) - U_0^{-r}W_t^*(W_t - W_s)U_0^r \\ &= U_0^{-r}W_t^*W_sU_0^r - W_t^*W_s \\ &= \sum_{k=0}^{r-1} (U_0^{-k-1}W_t^*W_sU_0^{k+1} - U_0^{-k}W_t^*W_sU_0^k). \end{aligned}$$

Since

$$U_0^{-k-1}W_t^*W_sU_0^{k+1} - U_0^{-k}W_t^*W_sU_0^k = U_0^{-k-t-1} (TU^{t-s} - U^{t-s}T) U_0^{s+k},$$

we obtain

$$\begin{aligned} & W_t^*(W_t - W_s) - U_0^{-r}W_t^*(W_t - W_s)U_0^r \\ &= \sum_{k=0}^{r-1} U_0^{-k-t-1} (TU^{t-s} - U^{t-s}T) U_0^{s+k}. \end{aligned}$$

Since, by Lemma 3.3,  $s\text{-}\lim_{r \rightarrow \infty} U_0^{-r}W_t^*(W_t - W_s)U_0^r\psi = 0$ , we have

$$\begin{aligned} W_t^*(W_t - W_s)\psi &= \sum_{k=0}^{\infty} U_0^{-k-t-1} (TU^{t-s} - U^{t-s}T) U_0^{s+k}\psi \\ &= Z_{t,s}((U_0T)U^{t-s} - (U_0U^{t-s})T)\psi, \end{aligned}$$

where

$$Z_{t,s}(A) = \sum_{k=0}^{\infty} U_0^{-k-t}AU_0^{k+s}.$$

By Lemma 3.4 below, we know that

$$\begin{aligned} |\langle \psi, W_t^*(W_t - W_s)\psi \rangle| &\leq |\langle \psi, Z_{t,s}((U_0T)U^{t-s})\psi \rangle| \\ &\quad + |\langle \psi, Z_{t,s}(U_0U^{t-s})T\psi \rangle| \rightarrow 0, \quad t, s \rightarrow \infty. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.4.** Let  $Y \in \mathcal{T}_1$  and  $\{Q(t, s)\}$  be a family of bounded operators with  $\sup_{t,s} \|Q(t, s)\| < \infty$ . Then, for all  $\psi \in \mathcal{H}_{ac,0}$ ,

$$(1) \lim_{t,s \rightarrow \infty} \langle \psi, Z_{t,s}(YQ(t, s))\psi \rangle = 0;$$

$$(2) \lim_{t,s \rightarrow \infty} \langle \psi, Z_{t,s}(Q(t, s)Y)\psi \rangle = 0.$$

*Proof.* Let  $Y = \sum_{n=1}^{\infty} \lambda_n |\psi_n\rangle\langle\phi_n|$  be the canonical expansion of the compact operator  $Y$ . Since  $Y \in \mathcal{T}_1$ ,  $\sum_n \lambda_n < \infty$ . Then, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\langle \psi, Z_{t,s}(YQ(t, s))\psi \rangle| &\leq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \lambda_n |\langle U_0^{k+t}\psi, \psi_n \rangle \langle \phi_n, Q(t, s)U_0^{k+s}\psi \rangle| \\ &\leq I_1(t, s)^{1/2} \times I_2(t, s)^{1/2}, \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \lambda_n |\langle \psi_n, U_0^{k+t}\psi \rangle|^2, \\ I_2(t, s) &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \lambda_n |\langle Q(t, s)^*\phi_n, U_0^{k+s}\psi \rangle|^2. \end{aligned}$$

By Lemma 3.2, we have

$$I_2(t, s) \leq 2\pi \sup_{\lambda} G_{\psi}(\lambda)^2 \sup_{t,s} \|Q(t, s)\| \sum_n \lambda_n < \infty,$$

where we have used the fact that  $\phi_n$  is a normalized vector. Let  $u_k = \sum_{n=1}^{\infty} \lambda_n |\langle \psi_n, U_0^k\psi \rangle|^2$ . Then, similarly to the above, we observe that  $\{u_k\} \in \ell^1(\mathbb{Z})$ . Hence, we have

$$\lim_{t \rightarrow \infty} I_1(t) = \lim_{t \rightarrow \infty} \sum_{k=t}^{\infty} u_k = 0.$$

This proves (i). The same proof works for (ii). □

## 4 Asymptotic velocity

### 4.1 Proof of Theorem 2.2

Let

$$\mathcal{H}_0 = \bigcup_{m=0}^{\infty} \{\Psi \in \mathcal{H} \mid \Psi(x) = 0, |x| \geq m\}.$$

We use  $\mathcal{D}$  to denote a subspace of vectors  $\Psi \in \mathcal{H}$  whose Fourier transform  $\hat{\Psi}$  are differentiable in  $k$  with

$$\sup_{k \in [0, 2\pi)} \left\| \frac{d}{dk} \hat{\Psi}(k) \right\| < \infty.$$

Note that  $\mathcal{H}_0$  is a core for  $\hat{x}$ , and so is  $\mathcal{D}$ . Let  $D = \mathcal{F} \hat{x} \mathcal{F}^{-1}$ . Then, by direct calculation, we know that  $(D\hat{\Psi})(k) = i \frac{d}{dk} \hat{\Psi}(k)$  for  $\Psi \in \mathcal{D}$ . We prove the following theorem:

**Theorem 4.1.** Suppose that (A.2) holds. Then,

$$\text{s-}\lim_{t \rightarrow \infty} \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} = (\hat{v}_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (4.1)$$

*Proof.* For all  $\Psi \in \mathcal{H}$  and  $\epsilon > 0$ , there exists a vector  $\Psi_\epsilon \in \mathcal{D}$  such that  $\|\Psi - \Psi_\epsilon\| \leq \epsilon$ . Because, by the second resolvent identity,

$$\begin{aligned} & \left\| \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} \Psi - (\hat{v}_0 - z)^{-1} \Psi \right\| \\ & \leq \frac{2\epsilon}{|\text{Im}z|} + \left\| \left( \frac{\hat{x}_0(t)}{t} - z \right)^{-1} \Psi_\epsilon - (\hat{v}_0 - z)^{-1} \Psi_\epsilon \right\| \\ & \leq \frac{2\epsilon}{|\text{Im}z|} + \frac{1}{|\text{Im}z|} \left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) (\hat{v}_0 - z)^{-1} \Psi_\epsilon \right\|, \end{aligned}$$

it suffices to prove that

$$\lim_{t \rightarrow \infty} \left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) (\hat{v}_0 - z)^{-1} \Psi \right\| = 0, \quad \Psi \in \mathcal{D}.$$

Note that

$$(\hat{v}_0 - z)^{-1} = \mathcal{F}^{-1} \left( \int_{[0, 2\pi)}^{\oplus} dk \sum_{j=1,2} \left( \frac{i\lambda'_j(k)}{\lambda_j(k)} - z \right)^{-1} |u_j(k)\rangle \langle u_j(k)| \right) \mathcal{F}.$$

Since  $\lambda_j(k)$  is analytic and  $|\lambda_j(k)| = 1$ , we observe from (A.2) that  $(\hat{v}_0 - z)^{-1}$  leaves  $\mathcal{D}$  invariant. Hence, we only need to prove that

$$\lim_{t \rightarrow \infty} \left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\| = 0, \quad \Psi \in \mathcal{D}.$$

By direct calculation, we have

$$\begin{aligned}
& \left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\|^2 \\
&= \int_0^{2\pi} dk \left\| \sum_{j=1,2} \left( \frac{i\lambda_j'(k)}{\lambda_j(k)} \right) \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) - \hat{U}(k)^{-t} \frac{D}{t} \hat{U}(k)^t \hat{\Psi}(k) \right\|^2 \\
&= \int_0^{2\pi} \frac{dk}{t^2} \left\| \sum_{j=1,2} \lambda_j(k)^t \hat{U}(k)^{-t} \left( i \frac{d}{dk} \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) \right) \right\|^2.
\end{aligned}$$

By the definition of  $\mathcal{D}$  and (A.2), we know that

$$\sup_{k \in [0, 2\pi)} \left\| \left( i \frac{d}{dk} \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) \right) \right\| < \infty.$$

Hence, we have

$$\left\| \left( \hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\| = O(t^{-1}),$$

which completes the proof.  $\square$

## 4.2 Proof of Theorem 2.3

The proof falls naturally into two parts:

**Theorem 4.2.** Let  $U$  be a unitary operator on  $\mathcal{H}$ .  $\hat{x}(t) = U^{-t} \hat{x} U^t$  satisfies

$$\text{s-} \lim_{t \rightarrow \infty} \exp \left( i \xi \frac{\hat{x}(t)}{t} \right) \Pi_p(U) = \Pi_p(U), \quad \xi \in \mathbb{R}.$$

**Theorem 4.3.** Let  $U = SC$  and  $U_0 = SC_0$  satisfy (A.1) and (A.2). Then,

$$\text{s-} \lim_{t \rightarrow \infty} \exp \left( i \xi \frac{\hat{x}(t)}{t} \right) \Pi_{ac}(U) = \exp(i \xi \hat{v}_+) \Pi_{ac}(U), \quad \xi \in \mathbb{R}.$$

*Proof of Theorem 2.3.* By (A.3), we have

$$\begin{aligned}
\text{s-} \lim_{t \rightarrow \infty} \exp \left( i \xi \frac{\hat{x}(t)}{t} \right) &= \text{s-} \lim_{t \rightarrow \infty} \exp \left( i \xi \frac{\hat{x}(t)}{t} \right) (\Pi_p(U) + \Pi_p(U)) \\
&= \Pi_p(U) + \exp(i \xi \hat{v}_+) \Pi_{ac}(U).
\end{aligned}$$

This prove the theorem.  $\square$

It remains to prove Theorems 4.2 and 4.3.

*Proof of Theorem 4.2.* Let  $\mathcal{H}_p(U)$  be the direct sum of all eigenspaces of  $U$ . It suffices to prove that, for  $\Psi \in \mathcal{H}_p(U)$ ,

$$\text{s-}\lim_{t \rightarrow \infty} \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) \Psi = \Psi.$$

Let  $\lambda_n$  be the eigenvalues of  $U$  and take an ONB  $\{\eta_n\}_{n=1}^\infty$  of  $\mathcal{H}_p$  such that  $U\eta_n = \lambda_n\eta_n$ . We have  $\Pi_p(U) = \sum_n |\eta_n\rangle\langle\eta_n|$ . Let  $\epsilon > 0$ . For sufficiently large  $N$ ,  $\Psi_N = \sum_{n=1}^N \langle\eta_n, \Psi\rangle \eta_n$  satisfies  $\|\Psi - \Psi_N\| \leq \epsilon$ . Then,

$$\left\| \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) \Psi - \Psi \right\| \leq 2\epsilon + \left\| \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) \Psi_N - \Psi_N \right\|.$$

By direct calculation, we have

$$\begin{aligned} \left\| \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) \Psi_N - \Psi_N \right\| &= \left\| \left( \exp\left(i\xi \frac{\hat{x}}{t}\right) - 1 \right) U^t \Psi_N \right\| \\ &= \left\| \sum_{n=1}^N \lambda_n^t \langle\eta_n, \Psi\rangle \left( \exp\left(i\xi \frac{\hat{x}}{t}\right) - 1 \right) \eta_n \right\| \\ &\leq \sum_{n=1}^N |\langle\eta_n, \Psi\rangle| \left\| \left( \exp\left(i\xi \frac{\hat{x}}{t}\right) - 1 \right) \eta_n \right\|. \end{aligned} \quad (4.2)$$

Since  $\lim_{t \rightarrow \infty} |1 - e^{i\xi x/t}| = 0$ ,  $|1 - e^{i\xi x/t}| \leq 2$  and  $\sum_x \|\eta_n(x)\|_{\mathbb{C}^2}^2 = \|\eta_n\|^2 < \infty$ , we have

$$\lim_{t \rightarrow \infty} \left\| \left( \exp\left(i\xi \frac{\hat{x}}{t}\right) - 1 \right) \eta_n \right\|^2 = \lim_{t \rightarrow \infty} \sum_{x \in \mathbb{Z}} |e^{i\xi x/t} - 1|^2 \|\eta_n(x)\|_{\mathbb{C}^2}^2 = 0,$$

which, combined with (4.2), completes the proof.  $\square$

**Lemma 4.1.**  $[U_0, \exp(i\xi \hat{v}_0)] = 0$ .

*Proof.* By direct calculation, we have

$$\begin{aligned} [U_0, \exp(i\xi \hat{v}_0)] &= \text{s-}\lim_{t \rightarrow \infty} \left[ U_0, \exp\left(i\xi \frac{\hat{x}_0(t)}{t}\right) \right] \\ &= \text{s-}\lim_{t \rightarrow \infty} U_0 \left\{ \exp\left(i\xi \frac{\hat{x}_0(t)}{t}\right) - \exp\left(i\xi \frac{\hat{x}_0(t+1)}{t}\right) \right\} = 0. \end{aligned}$$

$\square$

*Proof of Theorem 4.3.* By (A.1) and (A.2), Theorems 2.1 and 2.2 hold. Then,  $W_+$  is a unitary operator from  $\mathcal{H}_{\text{ac}}(U_0)$  to  $\mathcal{H}_{\text{ac}}(U)$ . Hence, we have

$$\exp(i\xi\hat{v}_+)\Pi_{\text{ac}}(U) = W_+\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U).$$

By direct calculation, we observe that

$$\begin{aligned} I(t) &:= \exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Pi_{\text{ac}}(U) - \exp(i\xi\hat{v}_+)\Pi_{\text{ac}}(U) \\ &= W_t\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)W_t^*\Pi_{\text{ac}}(U) - W_+\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U) \\ &=: \sum_{j=1}^3 I_j(t), \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= W_t\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)(W_t^* - W_+^*)\Pi_{\text{ac}}(U), \\ I_2(t) &= W_t\left(\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right) - \exp(i\xi\hat{v}_0)\right)W_+^*\Pi_{\text{ac}}(U), \\ I_3(t) &= (W_t - W_+)\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U). \end{aligned}$$

Because  $W_t$  and  $\exp(i\xi\hat{x}_0(t)/t)$  are uniformly bounded, we know from Theorems 2.1 and 2.2 that  $\text{s-lim}_{t\rightarrow\infty} I_1(t) = \text{s-lim}_{t\rightarrow\infty} I_2(t) = 0$ . Hence, we have

$$\begin{aligned} I(t) &= (W_t - W_+)\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U) + o(1) \\ &= (W_t - W_+)\Pi_{\text{ac}}(U_0)\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U) \\ &\quad + (W_t - W_+)[\exp(i\xi\hat{v}_0), \Pi_{\text{ac}}(U_0)]W_+^*\Pi_{\text{ac}}(U) + o(1), \end{aligned}$$

where we have used the fact that  $\text{Ran}W_+^* = \mathcal{H}_{\text{ac}}(U_0)$ . Since, by Lemma 4.1,  $[\exp(i\xi\hat{v}_0), \Pi_{\text{ac}}(U_0)] = 0$ , we obtain from Theorem 2.1, that  $\text{s-lim}_{t\rightarrow\infty} I(t) = 0$ . This completes the proof.  $\square$

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